ELSEVIER
Journal of Computational and Applied Mathematics 144 (2002) 277–289
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Piecewise algebraic curve ☆

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Received 10 January 2001; received in revised form 5 June 2001

Abstract

A piecewise algebraic curve is defined by a bivariate spline function. Using the techniques of the *B-net form* of bivariate splines function, discriminant sequence of polynomial (cf. Yang Lu et al. (Sci. China Ser. E 39(6) (1996) 628) and Yang Lu et al. (Nonlinear Algebraic Equation System and Automated Theorem Proving, Shanghai Scientific and Technological Education Publishing House, Shanghai, 1996)) and the number of sign changes in the sequence of coefficients of the highest degree terms of sturm sequence, we determine the number of real intersection points of two piecewise algebraic curves whose common points are finite. A lower bound of the number of real intersection points is given in terms of the method of rotation degree of vector field. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 41A15; 67D07; 65F10

Keywords: Number of real intersection points; Discriminant sequence; Rotation degree

1. Introduction

Let $\Delta = \{T^{[1]}, \dots, T^{[N]}\}$ be a regular triangulation of a simply connected polygonal domain Ω in R^2. Given integer r and d with $d \geqslant r+1$, we denote by $S_d^r(\Delta) = \{s \in C^r(\Omega)\colon s|_T \in \Pi_d \; for\ all\ T \in \Delta\}$ the space of bivariate splines of degree d and smoothness r (with respect to Δ). Here $\Pi_d = span\{x^i y^j\colon i \geqslant 0,\ j \geqslant 0, i+j \leqslant d\}$ denotes the space of bivariate polynomials of total degree d. A piecewise algebraic curve is given by $s(x,y) = 0$, where $s(x,y) \in S_d^r(\Delta)$ is real coefficients.

Let f and g be in $R[x]$, the Sturm sequence of f and g is the sequence of polynomials $(f_0, \dots, f_l)$ defined as follows

$$f_0 = f, \quad f_1 = f'g$$

☆ Project is supported by The National Natural Science Foundation of China.

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PII: S0377-0427(01)00567-2

$f_i = f_{i-1}q_i - f_{i-2}$ with $q_i \in R[x]$ and $\deg(f_i) < \deg(f_{i-1})$ for $i = 2, \dots, k$, f_k is a greatest common divisor f and $f'g$.

$V(f, g; +\infty)$ (resp. $V(f, g; -\infty)$) denotes the number of sign changes in the sequence of coefficients of the highest degree terms of $(f_0, \dots, f_k)$ (resp. $(f_0(-x), \dots, f_k(-x))$).

Given a polynomial with general symbolic coefficients,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and $a_k = 0$ if $k < 0$ or $k > n$. The discrimination matrix of $f(x)$, denoted by $\mathrm{discr}(f)$, is defined as follows

$$c_{ij} = (n - \max(i, j))a_ia_j - \sum_{p=-1}^{\min(i,j)-1} (i + j - 2p)a_pa_{i+j-p},$$

$$\mathrm{discr}(f) = (c_{ij}) \quad i, j = 0, 1, \dots, n - 1.$$

The principal minors sequence of $\mathrm{discr}(f)$

$$D_1(f), \dots, D_n(f)$$

is called the discriminant sequence of $f(x)$.

We can determine the number of real roots of real coefficients polynomial by the following.

Proposition 1.1 (cf. Yang Lu et al. [5,6]). *Let $f(x)$ be a polynomial with degree n, $\sigma_0 = 1$, and $\sigma_i = D_{q_i}$ be the ith nonzero term in the discriminant sequence $\{D_1(f), \dots, D_n(f)\}$, $i = 1, \dots, k$. Let again $s_i = q_{i+1} - q_i - 1$, $q_0 = 0$. Then the number of distinct real roots of $f(x)$ is equal to*

$$\sum_{\substack{i=0 \\ s_i \text{ is even}}}^{k-1} (-1)^{s_i/2} \operatorname{sign}\left(\frac{\sigma_{i+1}}{\sigma_i}\right).$$

It is of theoretical and practical significance to determine the number of real intersection points of piecewise algebraic curves. Because it is hard to know whether the real intersection points of two given polynomials are in one cell, there are many essential difficulties for studying it. The Bezout number of C^0 splines function space was discussed in [3]. In a star region, an upper bound on the Bezout number of C^1 splines function space was given in [4].

In this paper, using discriminant sequences of polynomial and the number of sign changes $V(f, g; +\infty)$ and $V(f, g; -\infty)$, we obtain a lemma on the number of distinct real intersection points of algebraic curves. By virtue of *B-net form* of bivariate splines function, we determine the number of distinct real intersection points of piecewise algebraic curves. We give one of its lower bounds by applying the method of rotation degree of vector field.

2. Preliminaries and notations

Firstly, we introduce some notations.

Let $p(x, y) \in R[x, y]$, then $p(x, y) \in R[y][x]$. The discriminant sequence of $p(x, y)$ with respect to the variable x is denoted by

$$D_1(p, y), D_2(p, y), \dots, D_n(p, y), \tag{1}$$

where $D_i(p,y)\in R[y],\ i=1,\dots,n$.

Let

$$D_{(i,j)}(p,y) := \{D_r(p,y)\colon r=i,i+1,\dots,j\},\quad i\leqslant j.$$

$\gcd(g(y),D_{(i,j)}(p,y))$ denotes the greatest common divisor of $g(y)$, $D_i(p,y),D_{i+1}(p,y),\ \dots,D_j(p,y)$, where $g(y)\in R[y]$. Suppose that two bivariate polynomials $p_1(x,y)$ and $p_2(x,y)$ are of the following forms:

$$\begin{aligned} p_1(x,y) &= b_0x^n + b_1(y)x^{n-1}+,\dots,+b_n(y),\\ p_2(x,y) &= c_0x^m + c_1(y)x^{m-1}+,\dots,+c_m(y). \end{aligned} \tag{2}$$

where b_0,c_0 are constants and $b_0c_0\neq 0,\ b_i(y),c_j(y)\in R[y]$. If not, by the transformation, $u=x,\ v=y+ax$, where a is a suitable real constant, we can get the form (2).

Let

$$\mathrm{NMR}(p_1,p_2;x) := \frac{\mathrm{res}(p_1,p_2;x)}{\gcd(\mathrm{res}(p_1,p_2;x),\mathrm{res}'(p_1,p_2;x))}, \tag{3}$$

where $\mathrm{res}(p_1,p_2;x)$ is Sylvester resultant of p_1,p_2 with respect to the variable x and $\mathrm{res}'(p_1,p_2;x)$ is its derivative with respect to the variable y. It is well known that $\mathrm{NMR}(p_1,p_2;x)$ is a univariate polynomial and has no multiple roots.

3. The main lemma

Lemma 3.1. *If algebraic curves $p_1(x,y)=0$ and $p_2(x,y)=0$ have no common components, where $p_1(x,y)$ and $p_2(x,y)$ with degree n and m, respectively, have the form* (2) (*in Section* 2). *Let $p(x,y)=p_1^2+p_2^2$, $k=\max(n,m)$, $\{D_1(p,y),\dots,D_{2k}(p,y)\}$ be the discriminant sequence of $p(x,y)$ with respect to the variable x.* $\mathrm{Num}(\mathrm{NMR}(p_1,p_2;x))$ *denotes the number of distinct real roots of* $\mathrm{NMR}(p_1,p_2;x)$, (Section 2, (3)). *Then algebraic curves $p_1=0$ and $p_2=0$ have* $\mathrm{Num}(p_1,p_2)$ *distinct real intersection points in R^2, where*

$$\begin{aligned} \mathrm{Num}(p_1,p_2) = {} & \mathrm{Num}(\mathrm{NMR}(p_1,p_2;x))\\ & + \sum_{i=1}^{2k-1}\{V(\mathrm{NMR}(p_1,p_2;x),D_i(p,y)D_{i+1}(p,y);-\infty)\\ & - V(\mathrm{NMR}(p_1,p_2;x),D_i(p,y)D_{i+1}(p,y);+\infty)\}\\ & + \sum_{\substack{1<i\leqslant j\leqslant 2k-1\\ (j-i+1)\text{ is even}}} (-1)^{(j-i+1)/2}\\ & \times\{V(\gcd(\mathrm{NMR}(p_1,p_2;x),D_{(i,j)}(p,y)),D_{i-1}(p,y)D_{j+1}(p,y);-\infty)\\ & - V(\gcd(\mathrm{NMR}(p_1,p_2;x),D_{(i,j)}(p,y)),D_{i-1}(p,y)D_{j+1}(p,y);+\infty)\}. \end{aligned} \tag{4}$$

Proof. It is obvious that the number of distinct real solutions of the system of equations $p_1(x,y)=p_2(x,y)=0$ is equal to the number of distinct real solutions of the following system of equations

$$\mathrm{NMR}(p_1,p_2;x)=\frac{\mathrm{res}(p_1,p_2;x)}{\gcd(\mathrm{res}(p_1,p_2;x),\mathrm{res}'(p_1,p_2;x))}=0,$$

$$p=p_1^2+p_2^2=0. \tag{5}$$

Suppose that y_0 is a real root of $\mathrm{NMR}(p_1,p_2;x)$, then y_0 corresponds to a discriminant sequence of univariate polynomial $p(x,y_0)$ with respect to variable x

$$D_1(p,y_0),D_2(p,y_0),\ldots,D_{2k}(p,y_0), \tag{6}$$

where $D_1(p,y_0)=2kb_0^2$ or $2kc_0^2$.

Let $\sigma_i=D_{q_i}(p,y_0)$ be ith nonzero term in (6) $(i=1,\ldots,t)$. Denote by $\mathrm{Num}(p(x,y_0))$ the number of distinct real roots of polynomial $p(x,y_0)$. By Proposition 1.1, we have

$$\mathrm{Num}(p(x,y_0))=1+\sum_{\substack{i=1\\ s_i=q_{i+1}-q_i-1\\ s_i \text{ is even}}}^{t-1}(-1)^{s_i/2}\,\mathrm{sign}\left(\frac{\sigma_{i+1}}{\sigma_i}\right).$$

Hence

$$\begin{aligned}\mathrm{Num}(p(x,y_0))&=1+\sum_{\substack{i=1\\ s_i=q_{i+1}-q_i-1\\ s_i \text{ is even}}}^{t-1}(-1)^{s_i/2}\,\mathrm{sign}(\sigma_{i+1}\sigma_i)\\ &=1+\sum_{i=1}^{2k-1}\mathrm{sign}(D_i(p,y_0)D_{i+1}(p,y_0))\\ &\quad+\sum_{\substack{1<i\leqslant j\leqslant 2k-1\\ (j-i+1) \text{ is even}\\ D_i=D_{i+1}=,\ldots,D_j=0}}(-1)^{(j-i+1)/2}\,\mathrm{sign}(D_{i-1}(p,y_0)D_{j+1}(p,y_0)).\end{aligned} \tag{7}$$

Let $l=\mathrm{Num}(\mathrm{NMR}(p_1,p_2;x))$. And let y_0 take over all distinct real roots of polynomial $\mathrm{NMR}(p_1,p_2;x)$, $y_1,y_2,\ldots,y_l$. We obtain l series corresponding to discriminant sequences of univariate polynomial $p(x,y_i)$ with respect to variable x $(i=1,\ldots,l)$ as follows:

$$\begin{aligned}&D_1(p,y_1),D_2(p,y_1),\ldots,D_{2k}(p,y_1),\\ &D_1(p,y_2),D_2(p,y_2),\ldots,D_{2k}(p,y_2),\\ &\cdots\cdots\\ &D_1(p,y_l),D_2(p,y_l),\ldots,D_{2k}(p,y_l).\end{aligned} \tag{8}$$

It follows from (7), (8) that

$$\begin{aligned}\mathrm{Num}(p_1,p_2) &= \sum_{y_0\in Q}\mathrm{Num}(p(x,y_0))\\ &= \mathrm{Num}(\mathrm{NMR}(p_1,p_2;x)) + \sum_{i=1}^{2k-1}\left[\sum_{y_0\in Q, T_1(y_0)>0} 1 - \sum_{y_0\in Q, T_1(y_0)<0} 1\right]\\ &\quad + \sum_{\substack{1<i\leqslant j\leqslant 2k-1\\ (j-i+1)\text{ is even}}} (-1)^{(j-i+1)/2}\left[\sum_{s\in W, T_2(s)>0} 1 - \sum_{s\in W, T_2(s)<0} 1\right], \end{aligned}\tag{9}$$

where

$$Q=\{y_1,y_2,\ldots,y_l\},$$

$$W=\{y_0\colon\ \gcd(\mathrm{NMR}(p_1,p_2;x),D_{(i,j)}(p,y))|_{y_0}=0,\ y_0\in Q\},$$

$$T_1=D_i(p,y)D_{i+1}(p,y),$$

$$T_2=D_{i-1}(p,y)D_{j+1}(p,y).$$

By Sylvester's theorem [1], we get

$$\begin{aligned}\mathrm{Num}(p_1,p_2) &= \mathrm{Num}(\mathrm{NMR}(p_1,p_2;x))\\ &\quad + \sum_{i=1}^{2k-1}\{V(\mathrm{NMR}(p_1,p_2;x),D_i(p,y)D_{i+1}(p,y);-\infty)\\ &\qquad - V(\mathrm{NMR}(p_1,p_2;x),D_i(p,y)D_{i+1}(p,y);+\infty)\}\\ &\quad + \sum_{\substack{1<i\leqslant j\leqslant 2k-1\\ (j-i+1)\text{ is even}}} (-1)^{(j-i+1)/2}\\ &\quad \times\{V(\gcd(\mathrm{NMR}(p_1,p_2;x),D_{(i,j)}(p,y)),D_{i-1}(p,y)D_{j+1}(p,y);-\infty)\\ &\quad - V(\gcd(\mathrm{NMR}(p_1,p_2;x),D_{(i,j)}(p,y)),D_{i-1}(p,y)D_{j+1}(p,y);+\infty)\}.\end{aligned}$$

This proves Lemma 3.1. □

Now, we try to determine the number of distinct real intersection points by the method of Gröbner Basis.

$I=\langle p_1,p_2\rangle$ is an ideal generated by p_1,p_2. Using lex ordering ($x>y$), we can get reduced Gröbner Basis of the ideal I, $G=\{q_1,q_2,\ldots,q_s\}$, then $\mathrm{Num}(p_1,p_2)$ is finite if and only if there are two polynomials q_1 and q_s in G such that $\mathrm{LT}(q_1)=x^{n_1}$, $\mathrm{LT}(q_s)=y^{n_s}$(LT denotes polynomial leading term). Because G is reduced Gröbner Basis, the degree of every monomial in polynomials $(q_2,\ldots,q_{s-1})$ with respect to variable x (resp. y) is less than n_1(resp. n_s), and there are

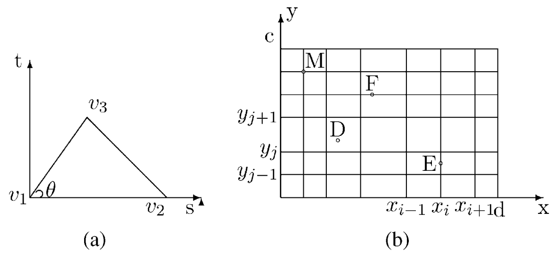

Fig. 1. (a) Selected coordinate system. (b) Place of intersection points.

no variable x in polynomial q_s. Hence, we can substitute $q_s/\gcd(q_s, q_s')$, $p = q_1^2 + q_2^2 +, \ldots, +q_s^2$ for $\mathrm{NMR}(p_1, p_2; x)$, $p = p_1^2 + p_2^2$ to compute $\mathrm{Num}(p_1, p_2)$ by using Lemma 3.1.

Next, we discuss lower bound on the number of real intersection points of algebraic curves in a triangle $\triangle(v_1, v_2, v_3)$ by the method of rotation degree of vector field.

We set coordinate system by selecting the longest edge of $\triangle(v_1, v_2, v_3)$ v_1v_2 as axis of abscissas, one of endpoints v_1 as origin (see Fig. 1a). In the coordinate system, two algebraic curves can be written as follows

$$p_1(s,t) = 0, \quad p_2(s,t) = 0. \tag{10}$$

Let $d = \tan\theta$, the straight line v_2v_3: $as + bt - c = 0$, $a, b > 0$. By the following transformation of variables:

$$x = \frac{t}{s} \quad y = as + bt \tag{11}$$

the interior points of $\triangle(v_1, v_2, v_3)$ are mapped into, one-to-one inclusion-reversing, open rectangle domain $(0, d) \times (0, c)$, the polynomial $p_1(s,t)$ and $p_2(s,t)$ are represented by

$$\begin{matrix} p_1(s,t) = \dfrac{1}{(a+bx)^n} f_1(x,y) \\ p_1(s,t) = \dfrac{1}{(a+bx)^m} f_2(x,y) \end{matrix} \quad (x,y) \in (0,d) \times (0,c), \tag{12}$$

where $f_1(x,y) \in \Pi_{2n}$, $f_2(x,y) \in \Pi_{2m}$. Hence, the number of real intersection points of $p_1(s,t) = 0$ and $p_2(s,t) = 0$ in the interior of $\triangle(v_1, v_2, v_3)$ is equal to the number of real intersection points of $f_1 = 0$ and $f_2 = 0$ in the rectangle $(0, d) \times (0, c)$, and the number of intersection points of $p_1(s,t) = 0$ and $p_2(s,t) = 0$ is finite if and only if the number of intersection points of $f_1 = 0$ and $f_2 = 0$ is finite on the whole R^2.

Let

$$\mathrm{NMR}(f_1, f_2; x) = \frac{\mathrm{res}(f_1, f_2; x)}{\gcd(\mathrm{res}(f_1, f_2; x), \mathrm{res}'(f_1, f_2; x))}$$

and

$$\mathrm{NMR}(f_1, f_2; y) = \frac{\mathrm{res}(f_1, f_2; y)}{\gcd(\mathrm{res}(f_1, f_2; y), \mathrm{res}'(f_1, f_2; y))}. \tag{13}$$

Suppose $g = a_n x^n +, \ldots, + a_1 x + a_0$, $a_n \neq 0$ and s_p satisfies the following Newton formula:

$$a_{p-1}s_1 + a_{p-2}s_2 +, \ldots, + a_0 s_p = -\, p a_p \quad p = 1, 2, \ldots, \tag{14}$$

where $a_k = 0$ if $p < 0$ or $p > n$. We can get s_p by solving linear system of equations (14).

Let

$$G_g = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \end{bmatrix}. \tag{15}$$

We have the following lemma.

Lemma 3.2. *Let $g = a_n x^n +, \ldots, + a_0$, $a_n \neq 0$, be a polynomial with real coefficient and with no multiple roots. Then the distance between any two roots of $g(x)$ is not less than*

$$\sqrt{\frac{\det G_g}{H_g^{(n+1)(n-2)}}}, \quad \textit{where } H_g = 2\max\left(1, \frac{1}{|a_n|}\sum_{i=1}^{n-1}|a_i|\right).$$

Proof. Let α_i $(i = 1, \ldots, n)$ be n distinct complex roots of the polynomial. It is well known that

$$\prod_{1 \leqslant j < i \leqslant n} (\alpha_i - \alpha_j)^2 = \Delta_n^2 = \Delta_n \Delta_n^{\mathrm{T}} = \det G_g, \tag{16}$$

$$|\alpha_i - \alpha_j| \leqslant 2\max\left(1, \frac{1}{|a_n|}\sum_{i=1}^{n-1}|a_i|\right), \tag{17}$$

where

$$\Delta_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{vmatrix}.$$

Let

$$L = \min(|\alpha_i - \alpha_j|\colon 1 \leqslant j < i \leqslant n), \quad H_g = 2\max\left(1, \frac{1}{|a_n|}\sum_{i=1}^{n-1}|a_i|\right). \tag{18}$$

$\prod_{1 \leqslant j < i \leqslant n} (\alpha_i - \alpha_j)^2$ contains $n(n-1)/2$ product terms, so we have

$$L^2 (H_g^{(n(n-1)/2)-1})^2 \geqslant G_g > 0,$$

hence

$$L \geqslant \sqrt{\frac{\det G_g}{H_g^{(n+1)(n-2)}}} > 0.$$

This proves Lemma 3.2.

We consider the plane autonomous system

$$\begin{aligned} \dot{x} &= f_1(x,y), \\ \dot{y} &= f_2(x,y). \end{aligned} \tag{19}$$

Then intersection points of $f_1 = 0$ and $f_2 = 0$ in the rectangle domain $(0,d) \times (0,c)$ change into singularities of the plane vector field (f_1, f_2) in the same domain, and the mapping is one-to-one including-reversing.

Suppose N is a simple closed curve on the plane, which does not pass through singularities of the plane vector field (f_1, f_2). It is well-known that the rotation degree of the closed curve N on the vector field (f_1, f_2), denoted by j, is equal to

$$j = \frac{1}{2\pi} \oint \mathrm{d} \arctan \frac{f_2}{f_1} = \frac{1}{2\pi} \oint_N \frac{f_1 \mathrm{d} f_2 - f_2 \mathrm{d} f_1}{f_1^2 + f_2^2}. \tag{20}$$

In order to work out another important lemma, we need the following auxiliary lemmas.

Lemma 3.3 (Ferdinand [2]). *If there are no intersection points of algebraic curves $f_1 = 0$ and $f_2 = 0$ in the domain D circled by closed curve N, then the rotation degree of the closed curve N on the vector field (f_1, f_2) is equal to zero.*

Lemma 3.4 (Ferdinand [2]). *If the rotation degree of closed curve N on vector field (f_1, f_2) is equal to nonzero, then there are intersection points of algebraic curves $f_1 = 0$ and $f_2 = 0$ in the domain D circled by the closed curve N.*

By (13), (14), (15) and Lemma (3.2), we can assume that

$$g_1(x) := \mathrm{NMR}(f_1, f_2; y) = a_u x^u +, \ldots, + a_1 x + a_0 \quad a_u \neq 0, \tag{21}$$

$$g_2(y) := \mathrm{NMR}(f_1, f_2; x) = b_v y^v +, \ldots, + b_1 y + b_0 \quad b_v \neq 0, \tag{22}$$

$$e_1 = \sqrt{\frac{\det G_{g_1}}{H_{g_1}^{(u+1)(u-2)}}}, \quad e_2 = \sqrt{\frac{det\, G_{g_2}}{H_{g_2}^{(v+1)(v-2)}}}. \tag{23}$$

Now, we present the second main lemma as follows.

Lemma 3.5. *If algebraic curves $p_1(s,t) = 0$ and $p_2(s,t) = 0$ have no common components. Let $x_i = ie_1$, $i = 0, 1, \ldots, n_1 - 1$, $x_{n_1} = d$, $y_j = je_2$, $j = 0, 1, \ldots, n_2 - 1$, $y_{n_2} = c$, be partition lines of the rectangle $[0,d] \times [0,c]$. where $n_1 = [d/e_1] + 1$, $n_2 = [c/e_2] + 1$. Then the number of distinct real* *intersection points of algebraic curves* $p_1 = 0$ *and* $p_2 = 0$ *in the interior of the triangle* $\triangle(v_1, v_2, v_3)$ *is not less than* $N_{\min}(p_1, p_2)$, *where*

$$N_{\min}(p_1,p_2)=\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}Z_{1ij}+\sum_{i=1}^{n_1-1}\sum_{j=0}^{n_2-1}Z_{2ij}+\sum_{j=1}^{n_2-1}\sum_{i=0}^{n_1-1}Z_{3ij}+\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}Z_{4ij}, \tag{24}$$

where

$$Z_{1ij}=|\mathrm{sign}[g_1(x_i)g_1(x_{i+1})g_2(y_j)g_2(y_{j+1})]|\left|\mathrm{sign}\left(\frac{1}{2\pi}\oint_{[x_i,x_{i+1}]\times[y_j,y_{j+1}]}\frac{f_1\mathrm{d}f_2-f_2\mathrm{d}f_1}{f_1^2+f_2^2}\right)\right|, \tag{25}$$

$$Z_{2ij}=(1-|\mathrm{sign}(g_1(x_i))|)|\mathrm{sign}[g_2(y_j)g_2(y_{j+1})]| \times\left|\mathrm{sign}\left(\frac{1}{2\pi}\oint_{[x_{i-1},x_{i+1}]\times[y_j,y_{j+1}]}\frac{f_1\mathrm{d}f_2-f_2\mathrm{d}f_1}{f_1^2+f_2^2}\right)\right|, \tag{26}$$

$$Z_{3ij}=(1-|\mathrm{sign}(g_2(y_j))|)|\mathrm{sign}[g_1(x_i)g_1(x_{i+1})]| \times\left|\mathrm{sign}\left(\frac{1}{2\pi}\oint_{[x_i,x_{i+1}]\times[y_{j-1},y_{j+1}]}\frac{f_1\mathrm{d}f_2-f_2\mathrm{d}f_1}{f_1^2+f_2^2}\right)\right|, \tag{27}$$

$$Z_{4ij}=1-\mathrm{sign}(|f_1(x_i,x_j)|+|f_2(x_i,x_j)|). \tag{28}$$

f_1, f_2 *above* (12), $g_1(x), g_2(y)$ *above* (21), (22).

Proof. According to (11) and (12), we need only to discuss the number of distinct real intersection points of $f_1 = 0$ and $f_2 = 0$ in the open rectangle $(0,d) \times (o,c)$.

We know, two algebraic curves $f_1 = 0$ and $f_2 = 0$ have intersection points in the open rectangle $(0,d) \times (o,c)$, the abscissas of these points are the real roots of polynomial $g_1(x)$ on the interval $(0,d)$ and their ordinate are the real roots of polynomial $g_2(y)$ on the interval $(0,c)$. By Lemma 3.2, (21), (22) and (23), we see that the distance between any two real roots of $g_1(x)$ (resp. $g_2(x)$) is not less than e_1 (resp. e_2). According to the partition of the rectangle $[0,d] \times [o,c]$, we can get a conclusion that there exists at most one intersection point on each closed small rectangle domain partitioned, and the place of the intersection point only have three cases. 1: in the interior of some small rectangle. 2: on the net line $x = x_i$ or $y = y_j$, $i = 1,\ldots,n_1 - 1$, $j = 1,\ldots,n_2 - 1$, excluding net points. 3: net point.

Case 1. The lower bound on the number of distinct real intersection points which lie in the interior of small rectangles partitioned, denoted by $N_{\min}^{(1)}$, is obtained.

According to the partition, if the intersection point D lies in the interior of some rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ (see Fig. 1b), they must satisfy the necessary condition as follows:

$$|\mathrm{sign}[g_1(x_i)g_1(x_{i+1})g_2(y_j)g_2(y_{j+1})]| = 1.$$

While the rotation degree of the vector field (f_1, f_2) on the boundary curve N of the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ is equal to

$$|\mathrm{sign}[g_1(x_i)g_1(x_{i+1})g_2(y_j)g_2(y_{j+1})]| \left(\frac{1}{2\pi} \oint_{[x_i, x_{i+1}]\times[y_j, y_{j+1}]} \frac{f_1 \mathrm{d} f_2 - f_2 \mathrm{d} f_1}{f_1^2 + f_2^2} \right).$$

By Lemmas 3.3 and 3.4, we know that there are at least Z_{1ij} (25) intersection points in the interior of the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. If $Z_{1ij} \neq 0$, then there exists only one intersection point of curves $f_1 = 0$ and $f_2 = 0$ in the interior of the rectangle. So, we can make a conclusion as follows:

$$N_{\min}^{(1)} = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} Z_{1ij}. \tag{29}$$

Case 2. The lower bound on the number of different real intersection points lying on the net lines $x = x_i$ or $y = y_j$, $i = 1, \ldots, n_1 - 1$, $j = 1, \ldots, n_2 - 1$, excluding net points, denoted by $N_{\min}^{(2)}$, is obtained.

The intersection point E lies on the net curve $x = x_i$ (see Fig. 1b). According to the partition, E must lie in the interior of some rectangle $[x_{i-1}, x_{i+1}] \times [y_j, y_{j+1}]$ and there is no other intersection point of $f_1 = 0$ and $f_2 = 0$ in it, and satisfy the necessary condition as follows:

$$(1 - |\mathrm{sign}(g_1(x_i))|)|\mathrm{sign}[g_2(y_j)g_2(y_{j+1})]| = 1.$$

Similar to case 1, there are at least Z_{2ij} (26) intersection point in the interior of the rectangle $[x_{i-1}, x_{i+1}] \times [y_j, y_{j+1}]$. Hence the smallest number of the intersection points lying on net line $x = x_i$ is equal to $\sum_{j=0}^{n_2-1} Z_{2ij}$, $i = 1, \ldots, n_1 - 1$.

Similarly, we can also be sure that the smallest number of the intersection points lying on net line $y = y_j$ is equal to $\sum_{i=0}^{n_1-1} Z_{3ij}$, $j = 1, \ldots, n_2 - 1$. So we have

$$N_{\min}^{(2)} = \sum_{i=1}^{n_1-1} \sum_{j=0}^{n_2-1} Z_{2ij} + \sum_{j=1}^{n_2-1} \sum_{i=0}^{n_1-1} Z_{3ij}. \tag{30}$$

Case 3. We can see easily that the number of distinct real intersection points lying on the net points is equal to $\sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} Z_{4ij}$.

So we complete the proof from Cases 1, 2 and 3. □

4. The main theorem

Let partition $\Delta = \{T^{[1]}, \ldots, T^{[N]}\}$, $T^{[l]} = \Delta(v_1^{[l]}, v_2^{[l]}, v_3^{[l]})$ be a triangle cell of the partition Δ, $l = 1, \ldots, N$. It is well known that for any given $s \in S_n^r(\Delta)$, the polynomials $p^{[l]} = s|_{T^{[l]}} \in \Pi_n$ can be represented in *B-net form* on the triangle cell $T^{[l]}$ as follows:

$$p^{[l]}(u_{l1}, u_{l2}, u_{l3}) = \sum_{|\lambda| = n} b_\lambda^{[l]} u_{l1}^{\lambda_1} u_{l2}^{\lambda_2} u_{l3}^{\lambda_3}, \tag{31}$$

where $b_\lambda^{[l]} = p_\lambda^{[l]} n!/(\lambda_1!\lambda_2!\lambda_3!)$, and (u_{l1}, u_{l2}, u_{l3}) is the barycentric coordinates of (x, y) on $T^{[l]}$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$, $\lambda_i \in \{0, 1, \ldots, n\}$, $p_\lambda^{[l]}$ are *Bezier* ordinates of $p^{[l]}$. By the transformation of variables as follows

$$u_{l1} = \left(\frac{2t_{l1}^2}{1 + t_{l1}^2 + t_{l2}^2}\right)^2, \quad u_{l2} = \left(\frac{2t_{l2}^2}{1 + t_{l1}^2 + t_{l2}^2}\right)^2, \quad u_{l3} = \left(\frac{t_{l1}^2 + t_{l2}^2 - 1}{1 + t_{l1}^2 + t_{l2}^2}\right)^2, \tag{32}$$

the polynomial $p^{[l]}(u_{l1}, u_{l2}, u_{l3})$ is represented by

$$p^{[l]}(u_{l1}, u_{l2}, u_{l3}) = \frac{4}{(1 + t_{l1}^2 + t_{l2}^2)^{2n}} p^{[l]^*}(t_{l1}, t_{l2}), \tag{33}$$

where

$$p^{[l]^*}(t_{l1}, t_{l2}) = \sum_{|\lambda|=n} b_\lambda^{[l]} t_{l1}^{2\lambda_1} t_{l2}^{2\lambda_2} (t_{l1}^2 + t_{l2}^2 - 1)^{2\lambda_3} \tag{34}$$

is defined on the whole R^2.

To simplify expression, we give some notations.

Γ_0 stands for all partition line segments, $\Gamma_1 = \{(v_3^{[l]} v_i^{[l]}) \subset T^{[l]}\colon l = 1, \ldots, N, i = 1, 2\}$, $\Gamma_2 = \{(v_1^{[l]} v_2^{[l]}) \subset T^{[l]}\colon l = 1, \ldots, N\}$, $\Gamma_3 = \{v_1^{[l]}, v_2^{[l]} \in T^{[l]}\colon l = 1, \ldots, N\}$, $\Gamma_4 = \{(v_3^{[l]} \in T^{[l]}\colon l = 1, \ldots, N\}$, where $(v_3^{[l]} v_i^{[l]})$ denotes partition line segment $v_3^{[l]} v_i^{[l]}$ excluding endpoints and if $v_3^{[l]} v_{i_1}^{[l]}$ and $v_3^{[k]} v_{i_2}^{[k]}$ stand for one line segment, $i_1, i_2 = 1, 2$, $l \neq k$, $l, k = 1, \ldots, N$. We still regard them as different line segment in Γ_1. Similarly in Γ_i, $i = 2, 3, 4$.

Obviously, the number of real intersection points of piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$ on Γ_0, Γ_1, and Γ_2 can be changed into the problem of the number of the real roots of one variable polynomial by making one of u_{l1}, u_{l2} and u_{l3} in (31) be equal to zero, we can use Sturm theorem and Proposition 1.1 to solve it. Whether piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$ pass through net points can determine the number of the intersection points of $s_1 = 0$ and $s_2 = 0$ on net points.

Theorem 4.1. *Let partition $\Delta = \{T^{[1]}, \ldots, T^{[N]}\}$, $T^{[l]} = \Delta(v_1^{[l]}, v_2^{[l]}, v_3^{[l]})$ be a triangle cell of the partition Δ, $l = 1, \ldots, N$. Suppose that $s_1 \in S_{n_1}^{r_1}(\Delta)$, $s_2 \in S_{n_2}^{r_2}(\Delta)$, the B-net form on $T^{[l]}$ of s_i is $p_i^{[l]}(u_{l1}, u_{l2}, u_{l3}) = s_i|_{T^{[l]}} = \sum_{|\lambda|=n_i} b_{i\lambda}^{[l]} u_{l1}^{\lambda_1} u_{l2}^{\lambda_2} u_{l3}^{\lambda_3}$, where $b_{i\lambda}^{[l]} = p_{i\lambda}^{[l]} n_i!/(\lambda_1!\lambda_2!\lambda_3!)$, $i = 1, 2$. If for any given $l \in \{1, \ldots, N\}$, polynomials whose restrict on the cell $T^{[l]}$ are $s_1|_{T^{[l]}}$ and $s_2|_{T^{[l]}}$ have no nontrivial common factor, then the number of distinct real intersection points of piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$ is equal to*

$$\frac{1}{4}\sum_{l=1}^{N} \mathrm{Num}(p_1^{[l]^*}, p_2^{[l]^*}) + \mathrm{Num}(s_1, s_2; \Gamma_0) - \tfrac{1}{2}\mathrm{Num}(s_1, s_2; \Gamma_1)$$
$$-\mathrm{Num}(s_1, s_2; \Gamma_2) - \tfrac{1}{2}\mathrm{Num}(s_1, s_2; \Gamma_3) - \tfrac{1}{4}\mathrm{Num}(s_1, s_2; \Gamma_4), \tag{35}$$

where

$$p_i^{[l]^*}(t_{l1}, t_{l2}) = \sum_{|\lambda|=n_i} b_{i\lambda}^{[l]} t_{l1}^{2\lambda_1} t_{l2}^{2\lambda_2} (t_{l1}^2 + t_{l2}^2 - 1)^{2\lambda_3}, \quad i = 1, 2, \ \mathrm{Num}(p_1^{[l]^*}, p_2^{[l]^*})$$

is computed by Lemma 3.1(4) *and* $\mathrm{Num}(s_1, s_2; \Gamma_j)$ *denotes the number of distinct real intersection points of piecewise algebraic curves $s_1 = 0$ and $s_2 = 0$ on Γ_j, $j = 0, 1, 2, 3, 4$.*

Proof. Let us first consider the number of distinct real intersection points of piecewise algebraic curves $s_1=0$ and $s_2=0$ in the interior of $T^{[l]}$.

System of equations

$$p_1^{[l]}(u_{l1},u_{l2},u_{l3})=\sum_{|\lambda|=n_1} b_{1\lambda}^{[l]}u_{l1}^{\lambda_1}u_{l2}^{\lambda_2}u_{l3}^{\lambda_3}=0,$$

$$p_2^{[l]}(u_{l1},u_{l2},u_{l3})=\sum_{|\lambda|=n_2} b_{2\lambda}^{[l]}u_{l1}^{\lambda_1}u_{l2}^{\lambda_2}u_{l3}^{\lambda_3}=0 \tag{36}$$

was transformed into (37) by the transformation of variables of (32)

$$p_1^{[l]^*}(t_{l1},t_{l2})=\sum_{|\lambda|=n_1} b_{1\lambda}^{[l]}t_{l1}^{2\lambda_1}t_{l2}^{2\lambda_2}(t_{l1}^2+t_{l2}^2-1)^{2\lambda_3}=0,$$

$$p_2^{[l]^*}(t_{l1},t_{l2})=\sum_{|\lambda|=n_2} b_{2\lambda}^{[l]}t_{l1}^{2\lambda_1}t_{l2}^{2\lambda_2}(t_{l1}^2+t_{l2}^2-1)^{2\lambda_3}=0. \tag{37}$$

According to (32), (33), (34), (36) and (37), we know that every intersection point of piecewise algebraic curves $s_1=0$ and $s_2=0$ in the interior of $T^{[l]}$ or on line segment $(v_1^{[l]}v_2^{[l]})$ corresponds to four solutions of system of equations (37), $(\pm t_{l1},\pm t_{l2})$, $t_{l1}t_{l2}\neq 0$, every intersection point on line segment $(v_3^{[l]}v_i^{[l]}]$ (excluding $v_3^{[l]}$) corresponds to two solutions of system of equations (37), $(\pm t_{l1},0)$ or $(o,\pm t_{l2})$, $i=1,2$. $t_{l1}t_{l2}\neq 0$, intersection point which pass through $v_3^{[l]}$ maps into zero solution. The reverse is also true. Hence the number of intersection points $s_1=0$ and $s_2=0$ in the interior of $T^{[l]}$ is equal to

$$\tfrac{1}{4}[\mathrm{Num}(p_1^{[l]^*},p_2^{[l]^*})-2(v_3v_1)-2(v_3v_1)-2v_1-2v_2-v_3], \tag{38}$$

where $\mathrm{Num}(p_1^{[l]^*},p_2^{[l]^*})$, which is the number of distinct real intersection points of curves $p_1^{[l]^*}=0$ and $p_2^{[l]^*}=0$ on the whole R^2, is computed by Lemma 3.1 (4), and for convenience, (v_iv_j) (resp.v_i) denotes the number of intersection points of piecewise algebraic curves $s_1=0$ and $s_2=0$ on line segment (v_iv_j) (resp. net v_i) $i,j=1,2,3$.

From above it follows that the number of distinct real intersection points of piecewise algebraic curves $s_1=0$ and $s_2=0$ in the interior of all cells is equal to

$$\frac{1}{4}\sum_{l=1}^{N}\mathrm{Num}(p_1^{[l]^*},p_2^{[l]^*})-\tfrac{1}{2}\mathrm{Num}(s_1,s_2;\Gamma_1)-\mathrm{Num}(s_1,s_2;\Gamma_2)$$
$$-\tfrac{1}{2}\mathrm{Num}(s_1,s_2;\Gamma_3)-\tfrac{1}{4}\mathrm{Num}(s_1,s_2;\Gamma_4). \tag{39}$$

Obviously, we immediately complete the proof from (39). □

Theorem 4.2. *Let partition $\Delta=\{T^{[1]},\ldots,T^{[N]}\}$, $T^{[l]}=\Delta(v_1^{[l]},v_2^{[l]},v_3^{[l]})$ be a triangle cell of the partition Δ, $l=1,\ldots,N$. And let $s_1\in S_{n_1}^{r_1}(\Delta)$, $s_2\in S_{n_2}^{r_2}(\Delta)$. If for any given $l\in\{1,\ldots,N\}$, polynomials whose restrict on the cell $T^{[l]}$ are $s_1|_{T^{[l]}}$ and $s_2i|_{T^{[l]}}$ have no nontrivial common factor, then the number of distinct real intersection points of piecewise algebraic curves $s_1=0$ and $s_2=0$*

is not less than

$$\sum_{l=1}^{N} N_{\min}(p_1^{[l]}, p_2^{[l]}) + \mathrm{Num}(s_1, s_2; \Gamma_0), \tag{40}$$

where polynomials $p_1^{[l]}$ and $p_2^{[l]}$ *are expression of* s_1 *and* s_2 (*respectively*) *on the triangle cell* $T^{[l]}$ *by selecting the coordinate system above* (10) (*see Fig.* 1a), $N_{\min}(p_1^{[l]}, p_2^{[l]})$ *is computed by Lemma* 3.5 (24), *and* $\mathrm{Num}(s_1, s_2; \Gamma_0)$ *denotes the number of distinct real intersection points of piecewise algebraic curves* $s_1 = 0$ *and* $s_2 = 0$ *on* Γ_0.

Proof. The proof is completed immediately from (10)–(12) and Lemma 3.5. □

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